Q. A hyperbola of the form $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$ has asymptotes with equations $y = \pm mx$ and passes thro' the point $(a, 0)$

a) Find an equation of the hyperbola in terms of x, y, a and m.

A point P on this hyperbola is equidistant from one of its asymptotes and the x-axis

b) Prove that, for all values of m, P lies on the curve with equation

$$(x^2 - y^2)^2 = 4x^2(x^2 - a^2)$$

---

a)

$$\frac{a^2}{\alpha^2} = 1 \Rightarrow \alpha^2 = a^2.$$

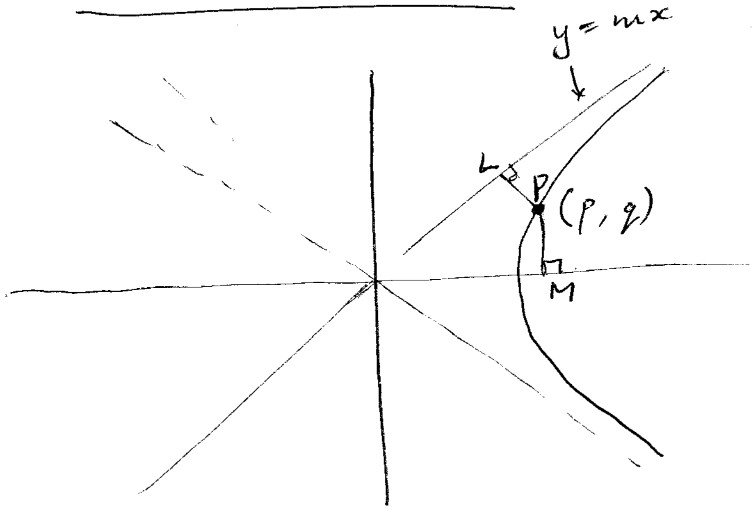
$$\frac{1}{\alpha} = \pm \frac{m}{\beta}$$

$$\beta = \pm \alpha m$$

$$\beta^2 = \alpha^2 m^2 \Rightarrow \beta^2 = a^2 m^2$$

$$\therefore \frac{x^2}{a^2} - \frac{y^2}{a^2 m^2} = 1 \Rightarrow m^2 x^2 - y^2 = a^2 m^2$$

b)



Eqn of LP: $y = -\frac{1}{m}x + c$

$$q = -\frac{1}{m}p + c$$

$$c = q + \frac{1}{m}p$$

$$\therefore y = -\frac{1}{m}x + q + \frac{1}{m}p$$

Coords of L : $mx + \frac{1}{m}x = q + \frac{1}{m}p$

$$m^2x + x = mq + p$$

$$\therefore\ x = \frac{mq+p}{1+m^2}$$

$$y = \frac{m}{1+m^2}(mq+p)$$

Distance LP^2;

$$\left[\frac{mq+p}{1+m^2} - p\right]^2 + \left[\frac{m}{1+m^2}(mq+p) - q\right]^2$$

$$= \frac{[mp-q]^2}{1+m^2}$$

<u>$LP = PM$</u>

$$\therefore\ \frac{m^2p^2 - 2mpq + q^2}{1+m^2} = q^2$$

$$m^2p^2 - 2mpq = m^2q^2$$

$$m(p^2-q^2) = 2pq$$

$$m^2(p^2-q^2)^2 = 4p^2q^2$$

$$m^2(p^2-q^2)^2 = 4p^2(m^2p^2 - m^2a^2)$$

$$\therefore\ (p^2-q^2)^2 = 4p^2(p^2-a^2) \qquad \text{(proved)}$$